# High temperature behavior of a two-parameter deformed quantum group fermion gas

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We discuss a two-parameter deformed quantum group fermion gas with $SU_{q/p}(2)$ symmetry. In order to obtain the role of the deformation parameters $(q,p)$ on the thermodynamics of the system, we calculate several thermodynamical functions and investigate the high temperature behavior of a $SU_{q/p}(2)$ fermion gas through a $SU_{q/p}(2)$-invariant fermionic Hamiltonian. However, the ordinary fermion gas results can be obtained by applying the limit $q=p=1$.

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## I. INTRODUCTION

Quantum groups and quantum algebras are generalizations of usual Lie groups and Lie algebras by some particular deformation parameters [1]. They have most popularly found application in a wide spectrum of research covering formulations of noncommutative geometry [2,3], exactly solvable statistical models [4], and two-dimensional conformal field theories [5]. Furthermore, statistical and thermodynamical consequences of studying $q$-deformed physical systems have intensively been investigated in the literature [6–9]. It should be mentioned that some possible connections between generalized statistical mechanics and quantum groups have recently been developed by some researchers [10,11]. In the framework of $q$ bosons and similar operators called quons [12], some considerable investigations have been carried out for obtaining a possible violation of the Pauli exclusion principle [13] and also a possible relation to anyonic statistics [14].

In this paper, our aim is to study the thermodynamical properties of a fermionic gas having the symmetry of the quantum group $SU_{q/p}(2)$. To achieve this aim, we simply consider a fermionic Hamiltonian invariant under the quantum group $SU_{q/p}(2)$. This Hamiltonian is constructed from the operators generating a two-parameter deformed $SU_{q/p}(2)$-covariant fermion algebra that becomes the usual fermion algebra in the limit $q=p=1$. We study such a two-parameter deformed fermion model to investigate its high temperature (low density) behavior, namely, for $z=e^{\beta\mu}\ll 1$, where $\beta=1/k_BT$, $k_B$ is the Boltzmann's constant, and $\mu$ is the chemical potential.

The paper is organized as follows. In Sec. II, we review the general properties of the $SU_q(N)$ fermions and specialize for $N=2$ case. In Sec. III, we introduce our model defined by a $SU_{q/p}(2)$-invariant fermionic Hamiltonian. This leads to the discussion of thermodynamics of the model obtained via the grand partition function given in Sec. IV. For instance, we find the average number of particles and the pressure. Therefore, the equation of state is derived as a virial expansion in order to determine the role of the deformation parameters $q$ and $p$ on the system. In the last section, we give our conclusions.

## II. QUANTUM GROUP $SU_q(N)$ FERMIONS

The usual fermion oscillators satisfy the following anticommutation relations:

$$\psi_i\psi_j^+ + \psi_j^+\psi_i=\delta_{ij},$$
$$\psi_i\psi_j+\psi_j\psi_i=0, \tag{1}$$
$$\psi_i^+\psi_i=N_i \quad i,j=1,2,\dots,N,$$

where $\psi_i$ and $\psi_i^+$ are the fermionic annihilation and creation operators, respectively, and $N_i$ is the fermion number operator. These oscillators are invariant under $SU(N)$ transformations. The quantum group analogs of these relations are written as follows [15,7]:

$$\Psi_j\bar{\Psi}_i=\delta_{ij}-q^{-1}R_{kijl}\bar{\Psi}_l\Psi_k, \tag{2}$$

$$\Psi_l\Psi_k=-qR_{jikl}\Psi_j\Psi_{ij} \quad i,j=1,2,\dots,N, \tag{3}$$

where the $N^2\times N^2$ matrix $R_{jikl}$ [3] is

$$R_{jikl}=\delta_{jk}\delta_{il}(1+(q-1)\delta_{ij})+(q-q^{-1})\delta_{ik}\delta_{jl}\theta(j-i), \tag{4}$$

and the function $\theta(j-i)=1$ for $j>i$ and zero otherwise. Under the linear transformation

$$\Psi_i'=\sum_{j=1}^{N}T_{ij}\Psi_j, \tag{5}$$

where the matrix $T\in SU_q(N)$, the relations given in Eqs. (2) and (3) are covariant. The $SU_q(N)$ transformation matrix $T$ and the $R$ matrix satisfy the following relations [16]:

$$RT_1T_2=T_2T_1R, \tag{6}$$

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}, \tag{7}$$

where $T_1=T\otimes 1$, $T_2=1\otimes T\in V\otimes V$, and $(R_{23})_{ijk,i'j'k'}=\delta_{ii'}R_{jk,j'k'}\in V\otimes V\otimes V$.

It can be found from Eq. (6) through a unitary quantum group matrix [17]

 

$$T=\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

that

$$ab=qba, \quad ac=qca,$$
$$cd=qdc, \quad bd=qdb,$$
$$bc=cb, \quad ad-da=(q-q^{-1})bc, \tag{8}$$
$$\mathrm{Det}_q(T)=ad-qbc=1.$$

Requiring $T$ to be unitary leads to the adjoint matrix $\bar{T}$ given by

$$\bar{T}=\begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}$$

with $q\in\mathbf{R}$. In particular, for $N=2$, the simplest $\mathrm{SU}_q(2)$-covariant algebra generated by the quantum group fermions $\Psi_i$, $i=1,2$, is given by the following relations:

$$\Psi_1\bar{\Psi}_1+\bar{\Psi}_1\Psi_1=1, \tag{9}$$

$$\Psi_2\bar{\Psi}_2+\bar{\Psi}_2\Psi_2=1+(q^2-1)\bar{\Psi}_1\Psi_1, \tag{10}$$

$$\Psi_1\Psi_2=-q\Psi_2\Psi_1, \tag{11}$$

$$\Psi_1\bar{\Psi}_2=-q\bar{\Psi}_2\Psi_1, \tag{12}$$

$$\{\Psi_1,\Psi_1\}=0=\{\Psi_2,\Psi_2\}, \tag{13}$$

in which they become the usual fermion algebra in the limit $q=1$. However, we consider a different quantum group $\mathrm{SU}_{\tilde{q}}$-(2) fermions, where $\tilde{q}=q/p$. We should mention now that historically, the foundations of the two-parameter deformed quantum group invariant bosonic oscillator algebras were based on Refs. [18,19], whereas the two-parameter deformed quantum group invariant fermionic oscillator algebra has recently been realized [20]. The anticommutation relations generating the quantum group $\mathrm{SU}_{q/p}(2)$ fermions are defined by

$$\Psi_1\bar{\Psi}_1+\bar{\Psi}_1\Psi_1=p^{2N_2}, \tag{14}$$

$$\Psi_2\bar{\Psi}_2+p^2\bar{\Psi}_2\Psi_2=p^{2N_2}+(q^2-1)\bar{\Psi}_1\Psi_1, \tag{15}$$

$$\Psi_1\Psi_2=-qp^{-1}\Psi_2\Psi_1, \tag{16}$$

$$\bar{\Psi}_1\bar{\Psi}_2=-pq^{-1}\bar{\Psi}_2\bar{\Psi}_1, \tag{17}$$

$$\Psi_1\bar{\Psi}_2=-qp\bar{\Psi}_2\Psi_1, \tag{18}$$

$$\{\Psi_1,\Psi_1\}=0=\{\Psi_2,\Psi_2\}, \tag{19}$$

where $N_2$ is the fermionic number operator and $q,p\in\mathbf{R}$. Hereafter, we will consider $0<q<\infty$ and $0<p<\infty$. However, the usual quantum group $\mathrm{SU}_q(2)$ fermions can be recovered in the limit $p=1$ as defined by Eqs. (9)–(13). It is obvious from Eq. (19) that occupation numbers for $\mathrm{SU}_{q/p}(2)$-fermionic states are restricted to $m=0,1$.

### III. $\mathrm{SU}_{q/p}(2)$-FERMION MODEL

In this section, we wish to find a representation of $\Psi_i$ operators in terms of usual fermion operators $\psi_i$. To achieve this aim, we begin by considering the following Hamiltonian in terms of $\mathrm{SU}_{q/p}(2)$ generators for two different kinds of fermions with the same energy,

$$H_F=\sum_k \varepsilon_k(M_{1,k}+M_{2,k}), \tag{20}$$

where the operators $M_{1,k}$ and $M_{2,k}$ are defined by

$$M_{1,k}=\bar{\Psi}_{1,k}\Psi_{1,k}, \quad M_{2,k}=\bar{\Psi}_{2,k}\Psi_{2,k}, \tag{21}$$

$\varepsilon_k$ is the spectrum of energy, $k=0,1,2,\ldots,$ and $\{\bar{\Psi}_{i,k},\Psi_{j,k'}\}=0$, for $k\neq k'$. The operators $M_1$ and $M_2$ satisfy the following relations for a given $k$:

$$M_2\Psi_1-q^{-2}\Psi_1M_2=0,$$
$$M_1\Psi_2-p^{-2}\Psi_2M_1=0. \tag{22}$$

The normalized states of the above Hamiltonian can be built by applying the operators $\bar{\Psi}$ on the vacuum state $|0,0\rangle$ for a given $k$ as

$$\bar{\Psi}_2^n\bar{\Psi}_1^m|0,0\rangle, \quad n,m=0,1. \tag{23}$$

In order to express a new representation for $\Psi_i$ operators in terms of usual fermion operators $\psi_{i,k}$ and $\psi_{i,k}^+$ satisfying Eqs. (1), we exploit the following representations for a given $k$:

$$\Psi_1=\psi_1(1+(p-1)N_2), \quad \bar{\Psi}_1=\psi_1^+(1+(p-1)N_2), \tag{24}$$

$$\Psi_2=\psi_2(1+(q-1)N_1), \quad \bar{\Psi}_2=\psi_2^+(1+(q-1)N_1), \tag{25}$$

By virtue of this representation, we rewrite the Hamiltonian in Eq. (20) as

$$H_F=\sum_k \varepsilon_k(N_{1,k}+N_{2,k}+(q^2+p^2-2)N_{1,k}N_{2,k}), \tag{26}$$

where $N_{i,k}=\psi_{i,k}^+\psi_{i,k}$. When compared with the original Hamiltonian in Eq. (20), this representation leads to an interacting Hamiltonian for the system containing two different kinds of fermionic particles. The thermodynamics of such a system will be discussed in the next section.

We should add that the representations given in Eqs. (24) and (25) can be generalized for arbitrary $N$ case according to the following transformations:

$$\Psi_1=\psi_1\prod_{l=2}^{N}(1+(p-1)N_l),$$

$$\Psi_2=[1+(q-1)N_1]\psi_2\prod_{l=3}^{N}(1+(p-1)N_l),$$

$$\Psi_3=[1+(q-1)N_1][1+(q-1)N_2]\psi_3\prod_{l=4}^{N}(1+(p-1)N_l), \tag{27}$$

$$\dots,$$

$$\Psi_m=\prod_{l=1}^{N-1}(1+(q-1)N_l)\psi_m,$$

and similarly for the adjoint equations.

## IV. THERMODYNAMICS OF $SU_{q/p}(2)$-FERMION GAS

We now investigate the high temperature (low density) behavior of the $SU_{q/p}(2)$-fermion gas described by the Hamiltonian in Eq. (26). Let us consider the following grand partition function $Z_F$ of the system:

$$Z_F=\mathrm{Tr}\exp\left[-\beta\sum_k\epsilon_k(\bar{\Psi}_{1,k}\Psi_{1,k}+\bar{\Psi}_{2,k}\Psi_{2,k})\right]e^{\beta\mu(N_{1,k}+N_{2,k})}, \tag{28}$$

where the trace is taken over the states in Eq. (23). By using Eq. (26), this grand partition function becomes

$$Z_F=\prod_k\sum_{n_1=0}^{1}\sum_{n_2=0}^{1}\exp[-\beta\epsilon_k(n_1+n_2+(q^2+p^2-2)n_1n_2)]e^{\beta\mu(n_1+n_2)}, \tag{29}$$

$$=\prod_k(1+2e^{-\beta(\epsilon_k-p)}+e^{-\beta(\epsilon_k(q^2+p^2)-2\mu)}), \tag{30}$$

for which one can recover the square of usual fermion-type grand partition function in the limit $q=p=1$. Since we are investigating the high temperature behavior of the model, namely, the limit $z\ll 1$, in the three-dimensional momentum space, the grand partition function in Eq. (30) can be rewritten as

$$\ln Z_F=\frac{4\pi V}{h^3}\int_0^\infty p^2\ln(1+2e^{-\beta(\epsilon-\mu)}+e^{-\beta(\epsilon(q^2+p^2)-2\mu)})\times dp, \tag{31}$$

which can be expanded in the first three terms as follows

$$\ln Z_F=\frac{4\pi V}{h^3}\int_0^\infty p^2\left[2e^{-\beta\epsilon}z+\frac{z^2}{2!}(2e^{-\beta\epsilon(q^2+p^2)}-4e^{-2\beta\epsilon})+\frac{z^3}{3!}(16e^{-3\beta\epsilon}-12e^{-\beta\epsilon(1+(q^2+p^2))})+\cdots\right]dp, \tag{32}$$

which gives the following relation by calculating the integral:

$$\ln Z_F=\frac{4\pi V}{h^3}\left[\frac{\sqrt{\pi}}{2}\left(\frac{2m}{\beta}\right)^{3/2}z-\sqrt{\pi}\left(\frac{2m}{\beta}\right)^{3/2}z^2\xi'(q,p)+\sqrt{\pi}\left(\frac{2m}{\beta}\right)^{3/2}z^3\zeta'(q,p)+\cdots\right], \tag{33}$$

where the functions $\xi'(q,p)$ and $\zeta'(q,p)$ are

$$\xi'(q,p)=\frac{1}{4}\left[\frac{1}{\sqrt{2}}-\frac{1}{(q^2+p^2)^{3/2}}\right], \tag{34}$$

$$\zeta'(q,p)=\frac{1}{6}\left[\frac{4}{3\sqrt{3}}-\frac{3}{(1+q^2+p^2)^{3/2}}\right]. \tag{35}$$

One can calculate the average number of particles $\langle N\rangle$ by

$$\langle N\rangle=\frac{1}{\beta}\left(\frac{\partial\ln Z_F}{\partial\mu}\right)_{T,V}, \tag{36}$$

which leads to

$$\langle N\rangle=\frac{4\pi V}{h^3}\left[\frac{\sqrt{\pi}}{2}\left(\frac{2m}{\beta}\right)^{3/2}z-2\sqrt{\pi}\left(\frac{2m}{\beta}\right)^{3/2}\xi'(q,p)z^2+\cdots\right]. \tag{37}$$

By reverting this equation, we can find the fugacity as

$$z\approx\frac{1}{2}\left(\frac{h^2}{2\pi mkT}\right)^{3/2}\frac{\langle N\rangle}{V}+\xi'(q,p)\left(\frac{h^2}{2\pi mkT}\right)^3\left(\frac{\langle N\rangle}{V}\right)^2. \tag{38}$$

The pressure can also be calculated by

$$P=\frac{1}{\beta}\left(\frac{\partial\ln Z_F}{\partial V}\right)_{T,p}, \tag{39}$$

which gives

$$P=\frac{4\pi}{h^3\beta}\left[\frac{\sqrt{\pi}}{2}\left(\frac{2m}{\beta}\right)^{3/2}z-\sqrt{\pi}\left(\frac{2m}{\beta}\right)^{3/2}\xi'(q,p)z^2+\cdots\right]. \tag{40}$$

By using the above equations, the equation of state is derived as a virial expansion

$$PV=kT\langle N\rangle\left[1+\xi'(q,p)\left(\frac{h^2}{2\pi mkT}\right)^{3/2}\frac{\langle N\rangle}{V}+\cdots\right], \tag{41}$$

where the second virial coefficient $B_2(q,p)$ is

$$B_2(q,p)=\xi'(q,p)\left(\frac{h^2}{2\pi mkT}\right)^{3/2}. \tag{42}$$

Since we are dealing with the high temperature limit, we are particularly focusing our attention to this second virial coefficient and omitting the other terms. Obviously, the sign of

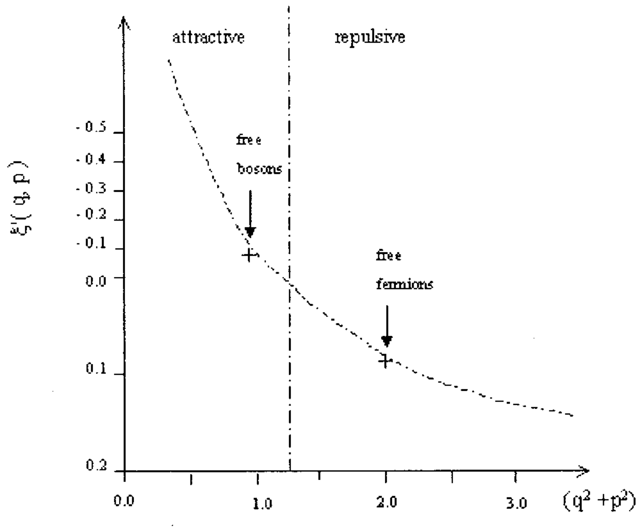


FIG. 1. The coefficient $\xi'(q,p)$ for the interval $0<(q^2+p^2)\leqslant 3$. The line at $(q^2+p^2)=1.26$ seperates the region between $\xi'(q,p)<0$ and $\xi'(q,p)>0$, which corresponds to bosonlike and fermionic behavior, respectively.

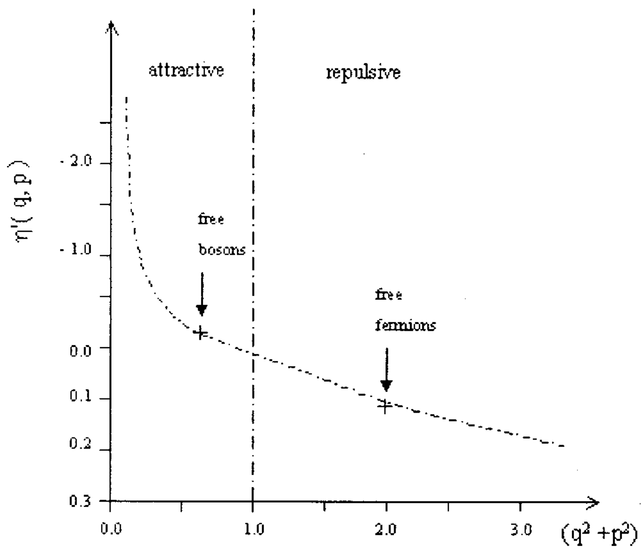


FIG. 2. The coefficient $\eta'(q,p)$ for the interval $0<(q^2+p^2)\leqslant 3$. The line at $(q^2+p^2)=1.0$ divides the region between $\eta'(q,p)<0$ and $\eta'(q,p)>0$, which corresponds to bosonlike and fermionic behavior, respectively.

the second virial coefficient depends on the values of the deformation parameters $q$ and $p$. Therefore, these parameters are responsible for the behavior of the present two-parameter fermion gas model. Figure 1 shows a graph of the coefficient $\xi'(q,p)$ as a function of the sum of the model parameters $q^2$ and $p^2$.

We now discuss some important limiting cases of the second virial coefficient $B_2(q,p)$ by means of the function $\xi'(q,p)$ given in Eq. (34). The function $\xi'(q,p)$ vanishes at $(q^2+p^2)\approx 1.26$, and, therefore, leads to an ideal gas result (up to the second virial coefficient). In the limit $q=p=1$, the coefficient $\xi'(1,1)=2^{-7/2}$, which describes a numerical factor in the second virial coefficient for a free fermion gas with two different kinds of fermionic particles. The function $\xi'(q,p)$ gets its highest value in the limit $q$ or $p\rightarrow\infty$ as $\xi'(q,p)\approx 0.18$. However, the free boson gas case [21] $\xi'(q,p)=-2^{-7/2}$ is reached at $(q^2+p^2)\approx 0.96$.

On the other hand, it is interesting to investigate whether a similar behavior could be found for the two-dimensional system by performing the same calculations. If one follows the same procedure as above, then the equation of state can be found as

$$PA=kT\langle N\rangle\left[1+\eta'(q,p)\left(\frac{h^2}{2\pi mkT}\right)\frac{\langle N\rangle}{A}+\cdots\right], \quad (43)$$

where $A$ is the surface confining the fermionic system and the function $\eta'(q,p)$ is

$$\eta'(q,p)=\frac{1}{4}\left[1-\frac{1}{(q^2+p^2)}\right] \quad (44)$$

The second virial coefficient $B_2(q,p)$ in Eq. (43) becomes

$$B_2(q,p)=\eta'(q,p)\left(\frac{h^2}{2\pi mkT}\right). \quad (45)$$

which clearly depends on the values of the function $\eta'(q,p)$. As is anticipated from this equation, the sign of the second virial coefficient changes depending on the values of the parameters $q$ and $p$. Indeed, this remarkable point is essential difference between the present two-parameter deformed fermionic gas model and the earlier one-parameter deformed fermionic gas model [8]. At this point, we need to add some remarks related to the second virial coefficient $B_2(q,p)$ by virtue of the function $\eta'(q,p)$ for the two-dimensional system. Figure 2 shows a graph of the coefficient $\eta'(q,p)$ as a function of a sum of the model parameters $q^2$ and $p^2$. The function $\eta'(q,p)$ vanishes at $(q^2+p^2)=1.0$, and, therefore, corresponds to an ideal gase case (up to the second virial coefficient). The free fermion gas result $\eta'(q,p)=2^{-3}$ with two different kinds of fermionic particles can be recovered in the limit $q=p=1$. The function $\eta'(q,p)$ gets its highest value in the limit $q$ or $p\rightarrow\infty$ as $\eta'(q,p)=0.25$. In the two-dimensional system, another interesting critical point is at $(q^2+p^2)\approx 0.67$ corresponding to $\eta'(q,p)=-2^{-3}$ such that this model behaves as a boson gas with two species in that values of the parameters $q$ and $p$ [21].

We wish to close this section by particularly discussing the third virial coefficient $B_3(q,p)$ for the two-dimensional system. Starting with Eq. (33), one can continue to calculate one-step further, all the procedure discussed above, and then the third virial coefficient $B_3(q,p)$ in Eq. (43) becomes

$$B_3(q,p)=\delta'(q,p)\left(\frac{h^2}{2\pi mkT}\right)^2, \quad (46)$$

where the function $\delta'(q,p)$ is

$$\delta'(q,p)=\frac{1}{2}\left[\frac{4}{9}-\frac{1}{(1+q^2+p^2)}\right]. \quad (47)$$

Although it seems to be of minor importance in the high temperature limit, when compared to the second virial coefficient $B_2(q,p)$ in Eq. (45), according to our point of view it would be significant to determine this third virial coefficient $B_3(q,p)$ in the two-parameter deformed fermion as well as boson models for both theoretical and experimental applications related to this field of research.

## V. CONCLUSIONS

In this paper, we studied the behavior of a two-parameter deformed quantum group fermionic gas $SU_{q/p}(2)$ at high temperatures. Starting with a $SU_{q/p}(2)$-invariant fermionic Hamiltonian, we calculated various thermodynamical functions via the grand partition function of the system, and consequently the equation of state is obtained as a virial expansion in the two- and three-dimensional space. Obviously, the free fermion gas results for a system containing two different kinds of fermionic particles can be found in the limit $q=p=1$.

We found that the sign of the second virial coefficient $B_2(q,p)$ depends on the parameters $q$ and $p$ in both two and three spatial dimensions. As shown in Fig. 1, the deformation parameters $q$ and $p$ interpolate between bosonlike and fermionic behaviors. Also, in three dimensions, we may remark that the $SU_{q/p}(2)$-fermion model exhibits an interpolation between attractive $[(q^2+p^2)<1.26]$ and repulsive $[(q^2+p^2)>1.26]$ systems.

The quantum group $SU_{q/p}(2)$-fermion model in two spatial dimensions has a crucial behavior through the parameters $q$ and $p$: This model exhibits an interpolation between attractive $[(q^2+p^2)<1.0]$ and repulsive $[(q^2+p^2)>1.0]$ systems including the free boson and fermion cases as shown in Figure 2. We have shown that this simple $SU_{q/p}(2)$-fermionic system describes such kinds of different systems spanned from fermionic to bosonic regions. Such a result indicates a similar physical behavior as seen in anyonic systems [22,23], whereas it is impossible to find such a behavior in the case of the one-parameter deformed quantum group fermionic gas in two dimensions [8,9], which correspond to the limit $p=1$ in the present model, except a difference point originating from $q$ and $q^{-1}$ factors between the two $SU_q(2)$-covariant oscillator algebras [Eq. (15)].

The low temperature behavior of the present two-parameter $SU_{q/p}(2)$-fermion model and generalization for higher $N$ fields will be the subject of another study that, hopefully, would give some new results.


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